# Math 564: Advance Analysis 1
## Lecture 22

**Lebesgue diff. for singular measures.** For each Borel measure $\mu$ on $\mathbb{R}^d$ that's $\underbrace{\text{finite on compact sets}}_{\text{locally finite}}$, if $\mu \perp \lambda$, then for $\lambda$-a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{\mu(\tilde{B}_r(x))}{\lambda(\tilde{B}_r(x))} = 0,$$

for any family $\{\tilde{B}_r(x)\}_{r>0}$ that shrinks $\lambda$-nicely to $x$.

**Proof.** It is enough to prove for balls because if $p \in (0,1)$ is s.t. $\lambda(\tilde{B}_r(x)) \geq p\,\lambda(B_r(x))$ $\forall r > 0$, then
$$\frac{\mu(\tilde{B}_r(x))}{\lambda(\tilde{B}_r(x))} \leq \frac{\mu(B_r(x))}{p \cdot \lambda(B_r(x))} \to 0 \text{ as } r \to 0.$$

Let $\mathbb{R}^d = X_\lambda \sqcup X_\mu$ be a Borel partition s.t. $X_\lambda$ is $\lambda$-conull and $X_\mu$ is $\mu$-conull.

It is enough to show that the sets
$$Z_\alpha := \left\{ x \in X_\lambda : \limsup_{r \to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} > \alpha \right\}$$
are $\lambda$-null because $\bigcup_{\alpha > 0} Z_\alpha = \bigcup_{n \in \mathbb{N}^+} Z_{1/n}$. We show that $\lambda(Z_\alpha) < \varepsilon$, $\forall \varepsilon > 0$.

By regularity of $\mu$, $\exists$ open $U \supseteq Z_\alpha$ s.t. $\mu(U) < \varepsilon$. $\forall x \in Z_\alpha$ $\exists$ open ball $B_x \subseteq U$ s.t. $\frac{1}{\alpha}\mu(B_x) > \lambda(B_x)$. Letting $U' := \bigcup_{x \in Z_\alpha} B_x$, we still have $U \supseteq U' \supseteq Z_\alpha$, so assume WLOG that $U = U'$. By Vitalli covering, $\forall$ positive $c < \lambda(U)$, $\exists$ fin. many pairwise disjoint balls $B_{x_1}, B_{x_2}, \dots, B_{x_n}$ s.t.
$$c \leq 3^d \sum_{i=1}^{n} \lambda(B_{x_i}) < 3^d \frac{1}{\alpha} \sum_{i=1}^{n} \mu(B_{x_i}) \leq 3^d \frac{1}{\alpha} \cdot \mu(U) < \frac{3^d}{\alpha} \cdot \varepsilon.$$

Thus, letting $c \nearrow \lambda(U)$, we get $\lambda(U) \leq \frac{3^d}{\alpha} \cdot \varepsilon$. Thus, $\lambda(Z_\alpha) \leq \frac{3^d}{\alpha} \cdot \varepsilon \to 0$ as $\varepsilon \to 0$. $\square$

**Cor.** Let $\mu$ be any loc. finite Borel measure on $\mathbb{R}^d$. Then for $\lambda$-a.e. $x \in \mathbb{R}^d$,

$$\lim_{r\to 0} \frac{\mu(\tilde{B}_r(x))}{\lambda(\tilde{B}_r(x))} = \frac{d\mu_{\ll\lambda}}{d\lambda}$$, where $\mu = \mu_{\ll\lambda} + \mu_{\perp\lambda}$ is Lebesgue dec of $\mu$ wrt $\lambda$, i.e. $\mu_{\ll\lambda} \ll \lambda$ and $\mu_{\perp\lambda} \perp \lambda$,

and $\{\tilde{B}_r(x)\}_{r>0}$ shrinks $\lambda$-nicely to $x$.

## Borel measures on $\mathbb{R}$ and the Fundamental Theorem of Calculus (FTC).

We saw in HW that if a measure $\mu$ on $\mathbb{R}$ is given by a continuously diff function $f$, i.e. $\mu((a,b]) = f(b) - f(a)$, then $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = f'$.
This happens in general:

**Cor.** Let $\mu$ be a loc. fin. Borel measure on $\mathbb{R}$ and let $f$ be an associated distribution function, i.e. $\mu((a,b]) = f(b) - f(a)$ $\forall\, a<b$.
Then $f'$ exists a.e. and is in $L^1_{loc}$. In fact, $f' = \frac{d\mu_{\ll\lambda}}{d\lambda}$. Thus:

(i) $\mu \ll \lambda$ if and only if the FTC holds for $f$, i.e. $\forall\, a<b$
$$f(b) - f(a) = \int_a^b f'\, d\lambda.$$

(ii) $\mu \perp \lambda$ if and only if $f' = 0$ a.e.

**Proof.** For each $x \in \mathbb{R}$, $f'(x) := \lim_{r\to 0} \frac{f(x+r) - f(x)}{r}$. To show that this limit exists it is enough to show that $\lim_{r\to 0^+} \frac{f(x+r) - f(x)}{r} = \frac{d\mu_{\ll\lambda}}{d\lambda}$ and $\lim_{r\to 0^+} \frac{f(x) - f(x-r)}{r} = \frac{d\mu_{\ll\lambda}}{d\lambda}$.

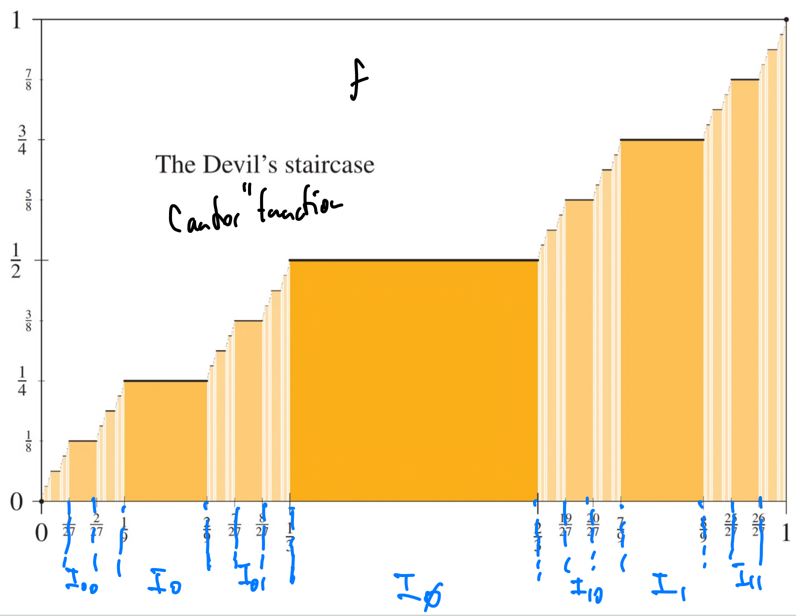
But the family $((x, x+r])_{r>0}$ shrinks $\lambda$-nicely to $x$ and the family $((x-r, x])_{r>0}$ also shrinks $\lambda$-nicely to $x$, so for $\lambda$-a.e. $x$, both of these limits exists and are equal to $d\mu_{\ll\lambda}/d\lambda$ by the previous corollary. Indeed, $\frac{f(x+r) - f(x)}{r} = \frac{\mu((x, x+r])}{\lambda((x,x+r])}$ and

$$\frac{f(x-r) - f(x)}{r} = \frac{\mu((x-r, x])}{\lambda((x-r, x])}.$$

For (i), we have $\mu \ll \lambda \iff \mu = \mu_{\ll\lambda} \iff \forall a<b \ \ \mu((a,b]) = \int_a^b \frac{d\mu}{d\lambda}\, d\lambda$
$\iff \forall a<b \ \ f(b)-f(a) = \int_a^b f'\, d\lambda$.

For (ii), $\mu \perp \lambda \iff \mu_{\ll\lambda} = 0 \iff \frac{d\mu_{\ll\lambda}}{d\lambda} = 0$ a.e. $\iff f' = 0$ a.e. $\square$

Example of $\mu \perp \lambda$: the devil's staircase. Let $C \subseteq [0,1]$ the standard Cantor set, $C := [0,1] \setminus U$, where $U = \bigsqcup_{s \in 2^{<\mathbb{N}}} I_s$, where each $I_s = (0.(2s)1, 0.(2s)2)$, where $(2s) := (2s_0)(2s_1)(2s_2)\dots(2s_{\mathrm{lh}(s)-1})$, i.e. replace all 1s in $s$ with 2s, where the numbers are in ternary rep. E.g., $(0.1, 0.2) = (\frac{1}{3}, \frac{2}{3})$, $(0.01, 0.02) = (\frac{1}{9}, \frac{2}{9})$, $(0.11, 0.12) = (\frac{4}{9}, \frac{2}{3})$.
Identifying $C$ with $2^{\mathbb{N}}$, let $\mu$ be the Bernoulli$(\frac{1}{2})$ measure on $C$.
What is the corresponding distribution function $f: [0,1] \to \mathbb{R}$, with $f(0)=0$?
We know $f$ is increasing and continuous because $\mu$ is atomless.




We also know that $f|_{I_s}$ is constant, in fact, $f|_{I_s} \equiv 0.s1$ in binary. Thus we know the def of $f$ on $U$, and one can verify that $f|_U$ is Lipschitz, hence uniformly continuous, hence extends continuously to the whole $[0,1]$.
The explicit definition of $f(x)$ is as follows: write $x$ in ternary, favouring 1, i.e. $0.021000\dots$ as opposed to $0.0202222\dots$ Then remove all digits after the first 1 (if such exists) and change all 2s to 1s. What we've obtained is the binary rep of $f(x)$.
Indeed $\forall x \in U$, $f'(x) = 0$ because $f$ is locally constant at $x$, and $[0,1]\setminus U$ is $\lambda$-null, hence $f' = 0$ a.e. yet $f$ is not constant.